



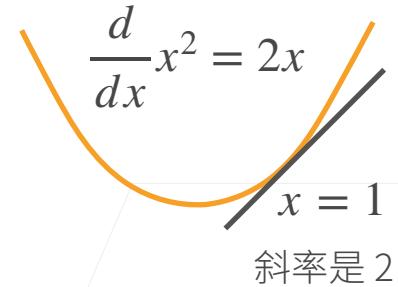
# 标量导数

$y$	$a$	$x^n$	$\exp(x)$	$\log(x)$	$\sin(x)$
$\frac{dy}{dx}$	0	$nx^{n-1}$	$\exp(x)$	$\frac{1}{x}$	$\cos(x)$

$a$  不是  $x$  的函数

$y$	$u + v$	$uv$	$y = f(u), u = g(x)$
$\frac{dy}{dx}$	$\frac{du}{dx} + \frac{dv}{dx}$	$\frac{du}{dx}v + \frac{dv}{dx}u$	$\frac{dy}{du} \frac{du}{dx}$

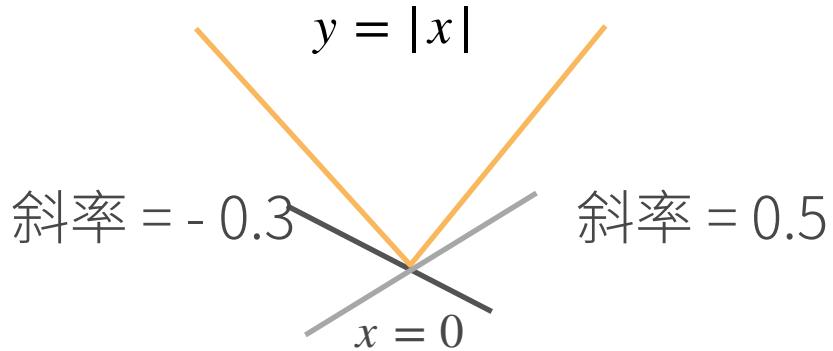
导数是切线的斜率





# 亚导数

- 将导数拓展到不可微的函数



另一个例子

$$\frac{\partial}{\partial x} \max(x, 0) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \\ a & \text{if } x = 0, \quad a \in [0, 1] \end{cases}$$

$$\frac{\partial |x|}{\partial x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ a & \text{if } x = 0, \quad a \in [-1, 1] \end{cases}$$



# 梯度

- 将导数拓展到向量

向量		
标量		
	$x$	$\mathbf{x}$
标量	$y$	$\frac{\partial y}{\partial x}$
向量	$\mathbf{y}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$



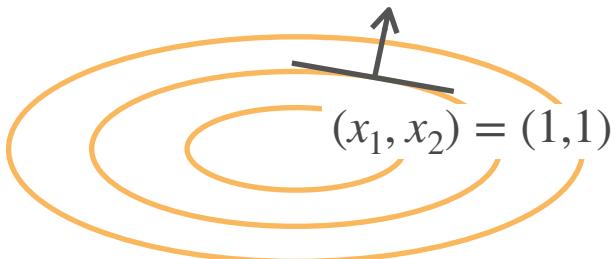
$\partial y / \partial \mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \frac{\partial y}{\partial \mathbf{x}} = \left[ \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n} \right]$$

$x$	$y$	$\frac{\partial y}{\partial x}$	$\mathbf{x}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$
			$\mathbf{y}$	$\frac{\partial \mathbf{y}}{\partial x}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$

$$\frac{\partial}{\partial \mathbf{x}} x_1^2 + 2x_2^2 = [2x_1, 4x_2]$$

方向  $(2, 4)$  跟等高线正交





# 样例

$y$	$a$	$au$	$\text{sum}(\mathbf{x})$	$\ \mathbf{x}\ ^2$
$\frac{\partial y}{\partial \mathbf{x}}$	$\mathbf{0}^T$	$a \frac{\partial u}{\partial \mathbf{x}}$	$\mathbf{1}^T$	$2\mathbf{x}^T$

$a$  is not a function of  $\mathbf{x}$

$\mathbf{0}$  and  $\mathbf{1}$  are vectors

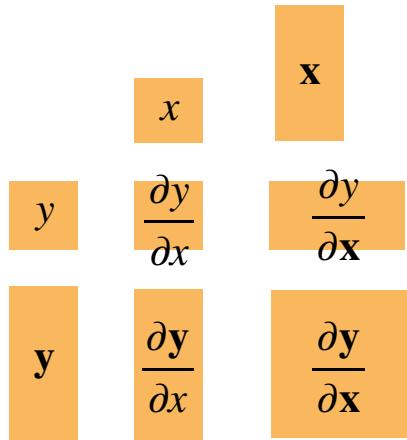
$y$	$u + v$	$uv$	$\langle \mathbf{u}, \mathbf{v} \rangle$
$\frac{\partial y}{\partial \mathbf{x}}$	$\frac{\partial u}{\partial \mathbf{x}} + \frac{\partial v}{\partial \mathbf{x}}$	$\frac{\partial u}{\partial \mathbf{x}}v + \frac{\partial v}{\partial \mathbf{x}}u$	$\mathbf{u}^T \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v}^T \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$



$\partial \mathbf{y} / \partial x$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\frac{\partial \mathbf{y}}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$



$\partial y / \partial \mathbf{x}$  是行向量,  $\partial \mathbf{y} / \partial x$  是列向量

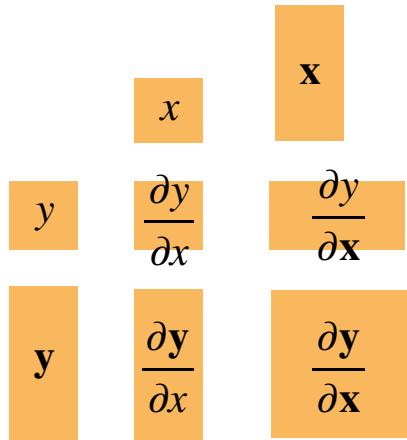
这个被称之为分子布局符号, 反过来的  
版本叫分母布局符号



$\partial \mathbf{y} / \partial \mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial \mathbf{x}} \\ \frac{\partial y_2}{\partial \mathbf{x}} \\ \vdots \\ \frac{\partial y_m}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \dots, \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \dots, \frac{\partial y_2}{\partial x_n} \\ \vdots \\ \frac{\partial y_m}{\partial x_1}, \frac{\partial y_m}{\partial x_2}, \dots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$





# 样例

$\mathbf{y}$	$\mathbf{a}$	$\mathbf{x}$	$\mathbf{Ax}$	$\mathbf{x}^T \mathbf{A}$
$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{A}$	$\mathbf{A}^T$

$$\mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y} \in \mathbb{R}^m, \quad \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \in \mathbb{R}^{m \times n}$$

$a$ ,  $\mathbf{a}$  and  $\mathbf{A}$  are not functions of  $\mathbf{x}$

$\mathbf{0}$  and  $\mathbf{I}$  are matrices

$\mathbf{y}$	$a\mathbf{u}$	$\mathbf{Au}$	$\mathbf{u} + \mathbf{v}$
$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$	$a \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\mathbf{A} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$



# 拓展到矩阵

	标量	向量	矩阵
标量	$x$ (1,)	$\mathbf{x}$ ( $n, 1$ )	$\mathbf{X}$ ( $n, k$ )
向量	$y$ (1,)	$\frac{\partial y}{\partial \mathbf{x}}$ (1,)	$\frac{\partial y}{\partial \mathbf{X}}$ ( $k, n$ )
矩阵	$\mathbf{y}$ ( $m, 1$ )	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ ( $m, 1$ )	$\frac{\partial \mathbf{y}}{\partial \mathbf{X}}$ ( $m, k, n$ )
	$\mathbf{Y}$ ( $m, l$ )	$\frac{\partial \mathbf{Y}}{\partial \mathbf{x}}$ ( $m, l$ )	$\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}$ ( $m, l, k, n$ )



# 链式法则和自动求导

